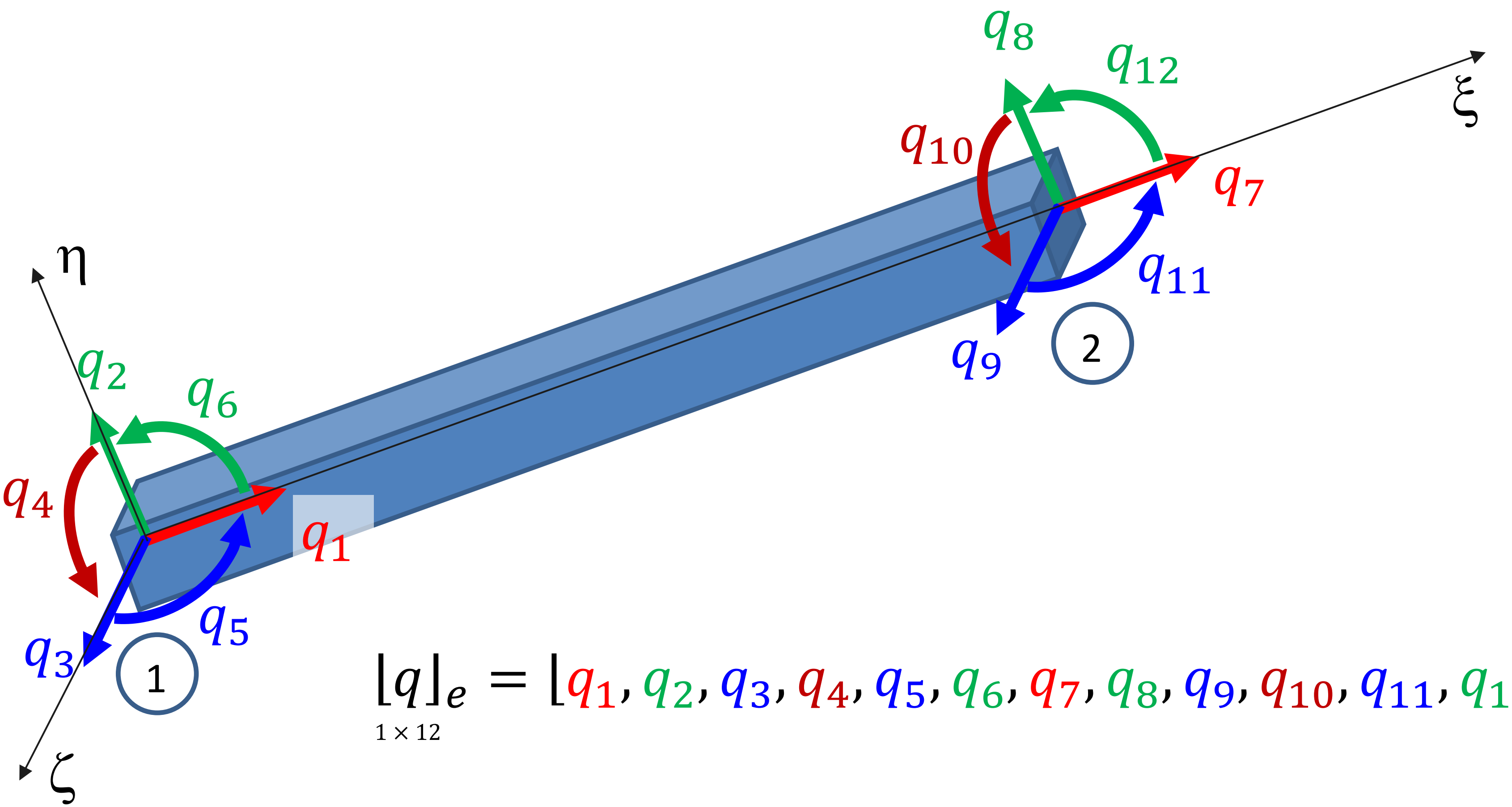
Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

Finite element method (FEM1)

Lecture 11A. 3D Frame element

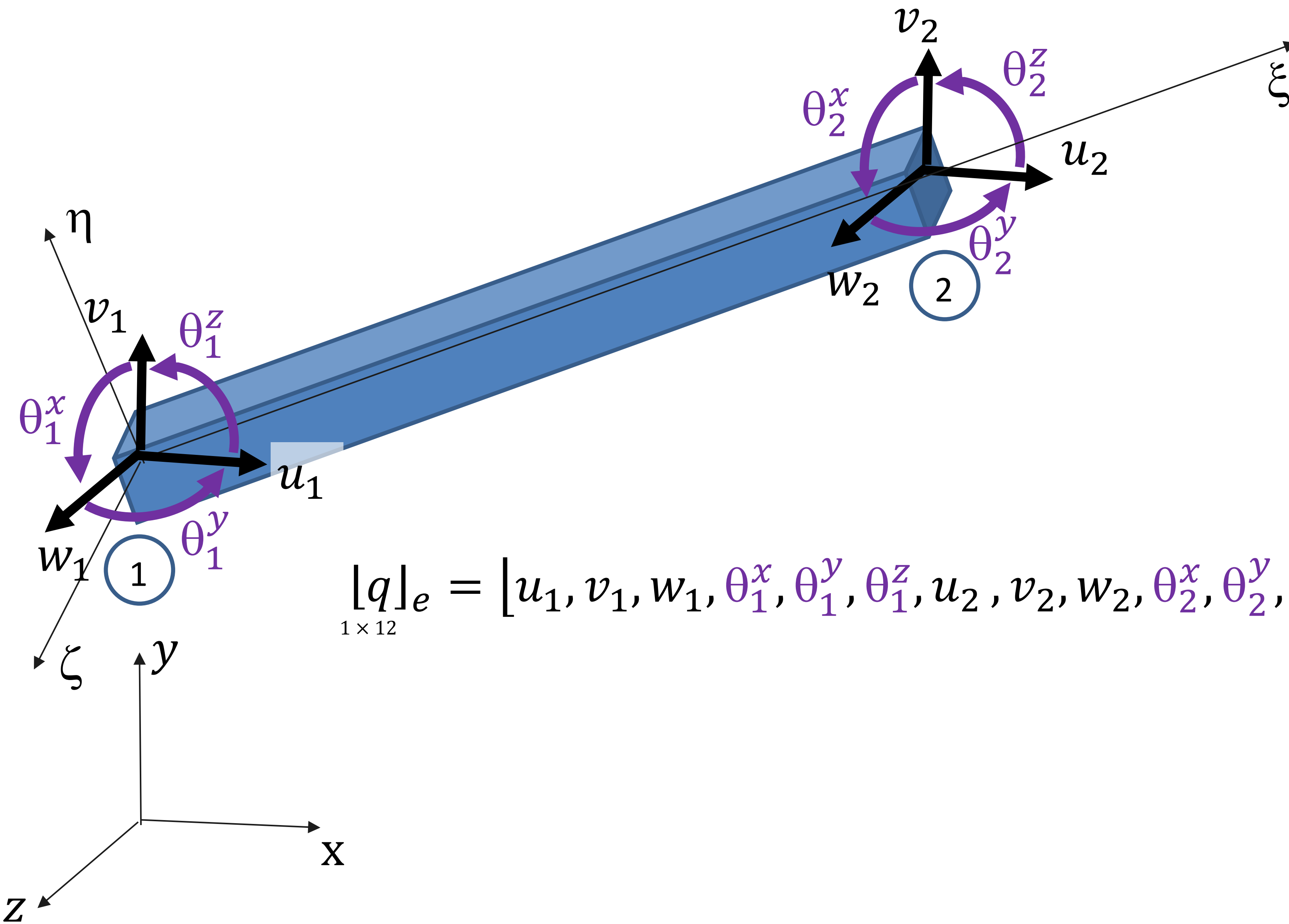
05.2025

3D Frame element in the local coordinate system (ξ, η, ζ)



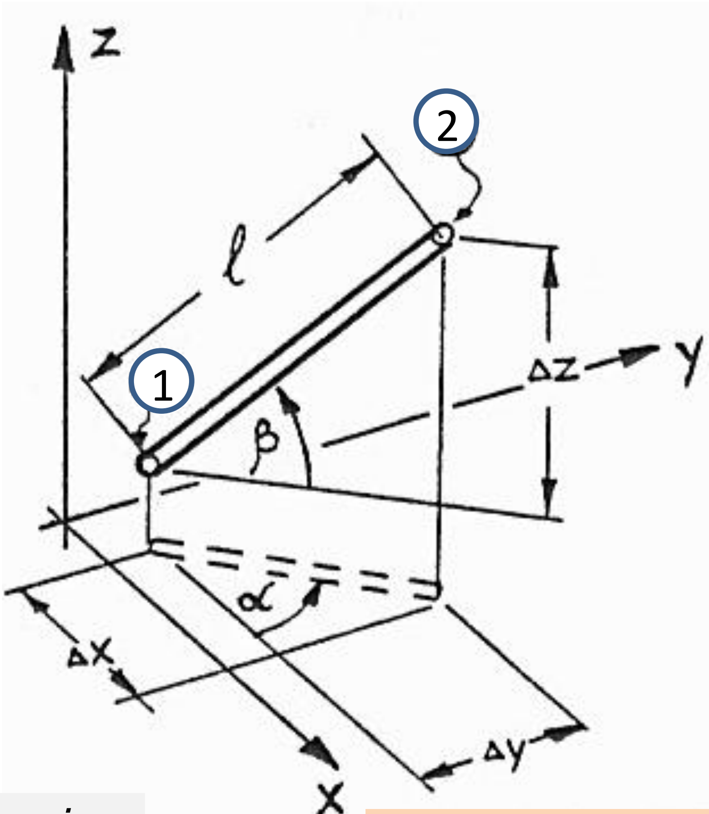
$$\underset{1\times 12}{\lfloor q \rfloor_e} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12} \rfloor$$

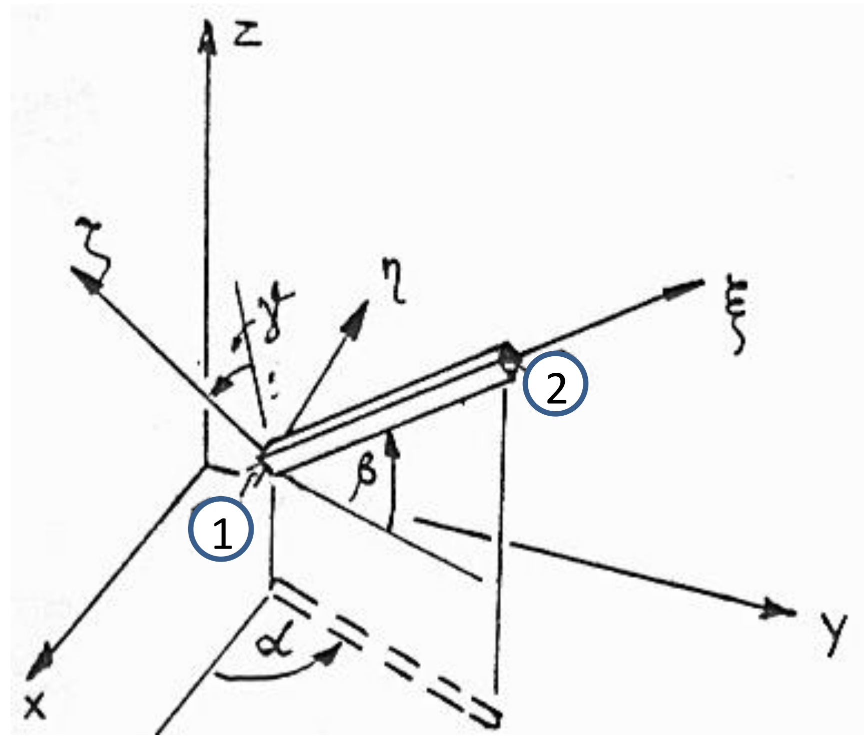
3D Frame element in the global coordinate system (x, y, z)



$$\underset{1\times 12}{\lfloor q \rfloor_e} = \lfloor u_1, v_1, w_1, \theta_1^x, \theta_1^y, \theta_1^z, u_2, v_2, w_2, \theta_2^x, \theta_2^y, \theta_2^z \rfloor$$

Transformation from the global coordinate system (x, y, z) to the local coordinate system (ξ, η, ζ)



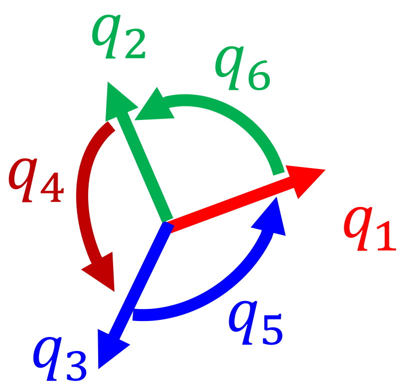


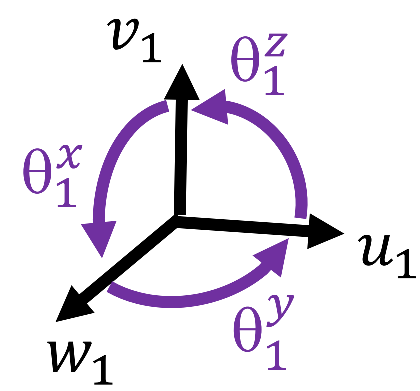
$s\alpha = sin\alpha$
$c\alpha = cos\alpha$
$s\beta = sin\beta$
$c\beta = cos\beta$
$s\gamma = sin\,\gamma$
$c\gamma = cos\,\gamma$

$$\lfloor \xi, \eta, \zeta \rfloor^T = [T]\lfloor x, y, z \rfloor^T$$

$$[T] = \begin{bmatrix} c\alpha \cdot c\beta & s\alpha \cdot c\beta & -s\beta \\ -(c\alpha \cdot s\beta \cdot s\gamma + s\alpha \cdot c\gamma) & -(s\alpha \cdot s\beta \cdot s\gamma - c\alpha \cdot c\gamma) & c\beta \cdot s\gamma \\ -(c\alpha \cdot s\beta \cdot c\gamma - s\alpha \cdot s\gamma) & -(s\alpha \cdot s\beta \cdot c\gamma + c\alpha \cdot s\gamma) & c\beta \cdot c\gamma \end{bmatrix}$$

Transformation of degrees of freedom from the global coordinate system (x, y, z) to local coordinate system (ξ, η, ζ)



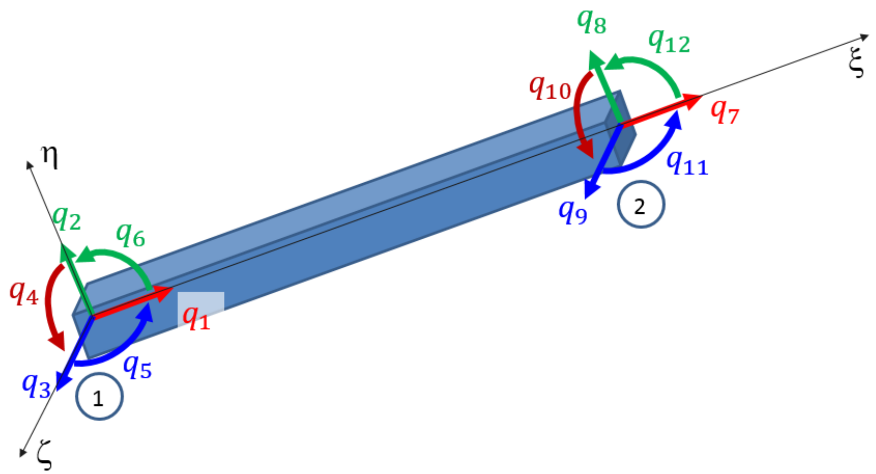


$$\{q\}_e = \begin{bmatrix} [T] & \\ & [T] \end{bmatrix} \{q_g\}_e = [T_r]\{q_g\}_e$$

$s\alpha = sin\alpha$
$c\alpha = cos\alpha$
$s\alpha = sin\beta$
$c\beta = cos\beta$
$s\gamma = sin\,\gamma$
$c\gamma = cos\,\gamma$

$$[T] = \begin{bmatrix} c\alpha \cdot c\beta & s\alpha \cdot c\beta & -s\beta \\ -(c\alpha \cdot s\beta \cdot s\gamma + s\alpha \cdot c\gamma) & -(s\alpha \cdot s\beta \cdot s\gamma - c\alpha \cdot c\gamma) & c\beta \cdot s\gamma \\ -(c\alpha \cdot s\beta \cdot c\gamma - s\alpha \cdot s\gamma) & -(s\alpha \cdot s\beta \cdot c\gamma + c\alpha \cdot s\gamma) & c\beta \cdot c\gamma \end{bmatrix}$$

Frame stiffness matrix components



$$[k_N] = \begin{bmatrix} \frac{AE}{l} & \frac{-AE}{l} \\ \frac{-AE}{l} & \frac{AE}{l} \end{bmatrix}$$

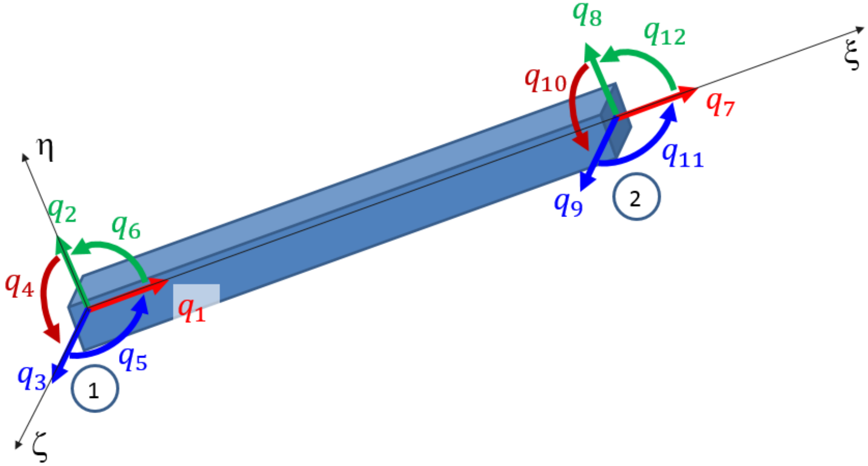
Stretching dependent on $[q_1, , q_7]$ degrees of freedom

$$[k_S] = \begin{bmatrix} \frac{GJ_S}{l} & \frac{-GJ_S}{l} \\ \frac{-GJ_S}{l} & \frac{GJ_S}{l} \end{bmatrix}$$

Twisting dependent on $[q_4, q_{10}]$ degrees of freedom

Frame stiffness matrix components

$$\left[k_{M_g}\eta\right]=\begin{bmatrix} \frac{12EJ_\eta}{l^3} & \frac{-6EJ_\eta}{l^2} & \frac{-12EJ_\eta}{l^3} & \frac{-6EJ_\eta}{l^2} \\ \frac{-6EJ_\eta}{l^2} & \frac{4EJ_\eta}{l} & \frac{6EJ_\eta}{l^2} & \frac{2EJ_\eta}{l} \\ \frac{-12EJ_\eta}{l^3} & \frac{6EJ_\eta}{l^2} & \frac{12EJ_\eta}{l^3} & \frac{6EJ_\eta}{l^2} \\ \frac{-6EJ_\eta}{l^2} & \frac{2EJ_\eta}{l} & \frac{6EJ_\eta}{l^2} & \frac{4EJ_\eta}{l} \end{bmatrix}$$

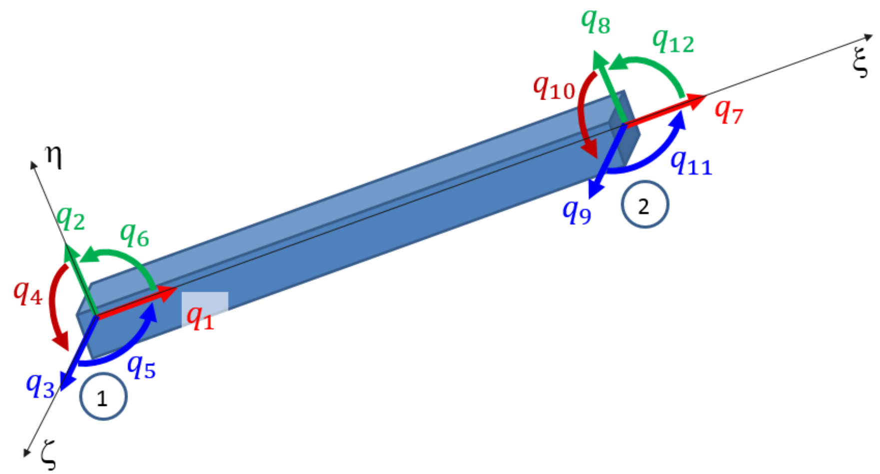


Bending about η axis dependent on $\lfloor q_3, q_5, q_9, q_{11} \rfloor$ degrees of freedom

$$\left[k_{M_g}\zeta\right]=\begin{bmatrix} \frac{12EJ_\zeta}{l^3} & \frac{6EJ_\zeta}{l^2} & \frac{-12EJ_\zeta}{l^3} & \frac{6EJ_\zeta}{l^2} \\ \frac{6EJ_\zeta}{l^2} & \frac{4EJ_\zeta}{l} & \frac{-6EJ_\zeta}{l^2} & \frac{2EJ_\zeta}{l} \\ \frac{-12EJ_\zeta}{l^3} & \frac{-6EJ_\zeta}{l^2} & \frac{12EJ_\zeta}{l^3} & \frac{-6EJ_\zeta}{l^2} \\ \frac{6EJ_\zeta}{l^2} & \frac{2EJ_\zeta}{l} & \frac{-6EJ_\zeta}{l^2} & \frac{4EJ_\zeta}{l} \end{bmatrix}$$

Bending about ζ axis dependent on $\lfloor q_2, q_6, q_8, q_{12} \rfloor$ degrees of freedom

Elastic strain energy of the frame element



Elastic strain energy of an element:

$$U_e = \frac{1}{2}\lfloor q \rfloor_e [k]_e \{q\}_e = \frac{1}{2}\lfloor q_g \rfloor_e [T_r]^T [k]_e [T_r]\{q_g\}_e ,$$

$$U_e = \frac{1}{2}\lfloor q_g \rfloor_e \left[k^g\right]_e \{q_g\}_e ,$$

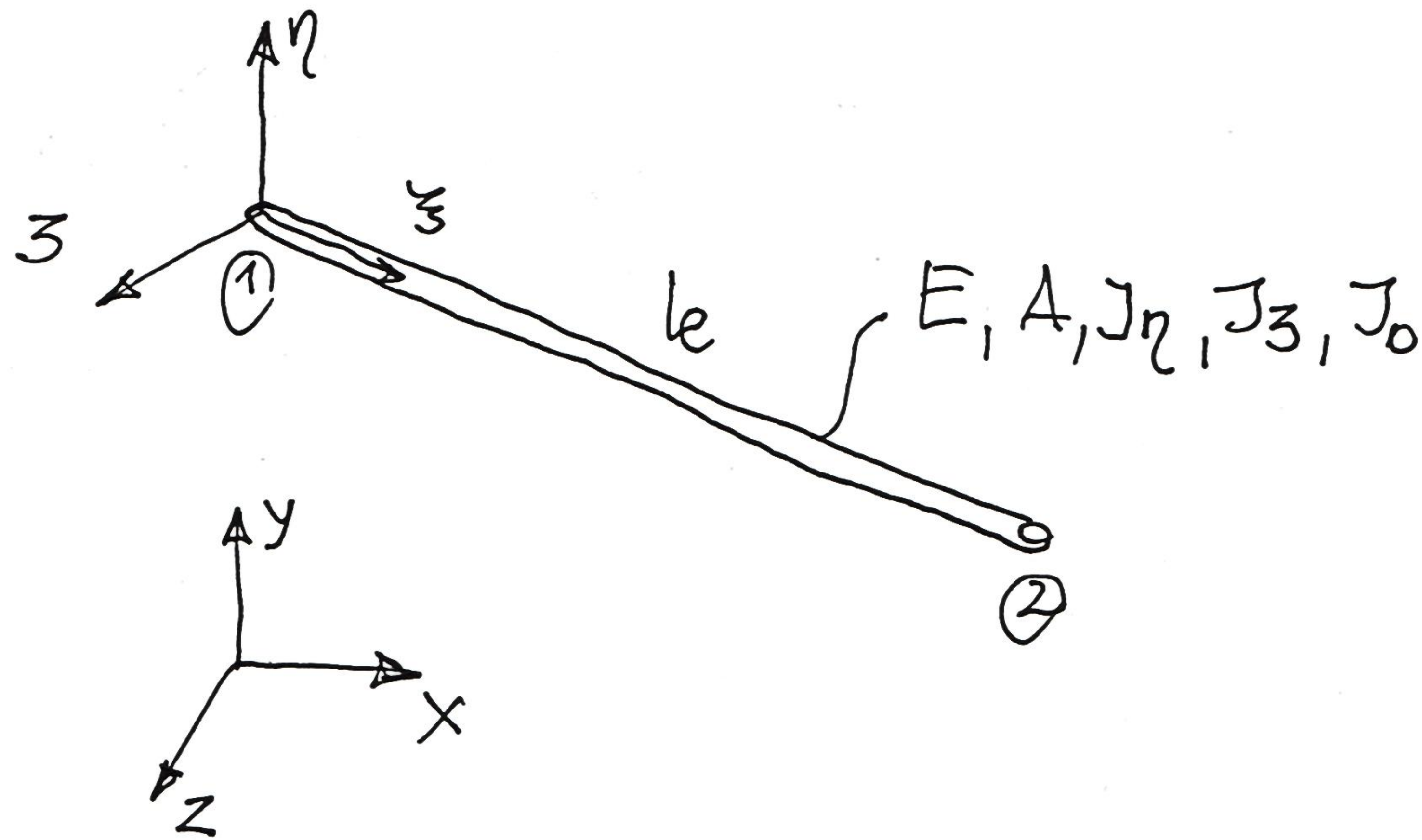
Element stiffness matrix:

$$\boxed{\left[k^g\right]_e = [T_r]^T [k]_e [T_r]}$$

3D Frame element stiffness matrix in local coordinate system (ξ, η, ζ)

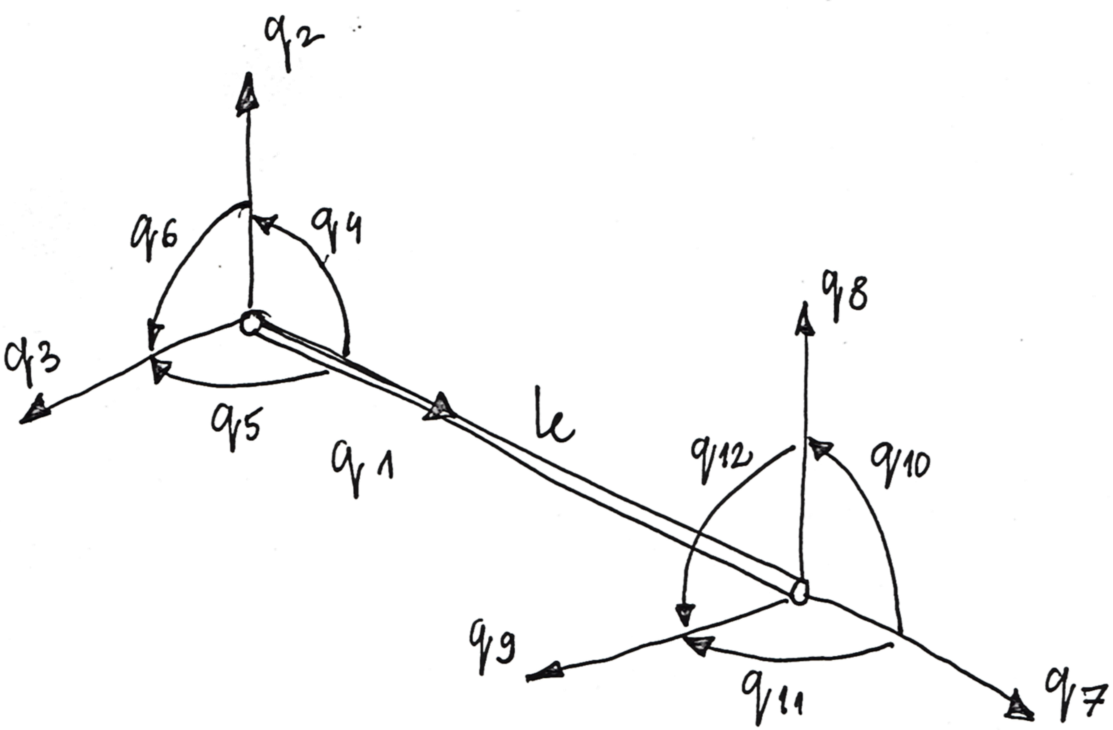
$$[k_e]=\begin{bmatrix}
\frac{AE}{l} & & & & & & \frac{-AE}{l} & & & & & \\
 & \frac{12EJ_\zeta}{l^3} & & & & \frac{6EJ_\zeta}{l^2} & & \frac{-12EJ_\zeta}{l^3} & & & & \frac{6EJ_\zeta}{l^2} \\
 & & \frac{12EJ_\eta}{l^3} & & \frac{-6EJ_\eta}{l^2} & & & & \frac{-12EJ_\eta}{l^3} & & \frac{-6EJ_\eta}{l^2} & \\
 & & & \frac{GJ_s}{l} & & & & & & \frac{-GJ_s}{l} & & \\
 & & \frac{-6EJ_\eta}{l^2} & & \frac{4EJ_\eta}{l} & & & & \frac{6EJ_\eta}{l^2} & & \frac{2EJ_\eta}{l} & \\
 & \frac{6EJ_\zeta}{l^2} & & & & \frac{4EJ_\zeta}{l} & & \frac{-6EJ_\zeta}{l^2} & & & & \frac{2EJ_\zeta}{l} \\
\frac{-AE}{l} & & & & & & \frac{AE}{l} & & & & & \\
 & \frac{-12EJ_\zeta}{l^3} & & & & \frac{-6EJ_\zeta}{l^2} & & \frac{12EJ_\zeta}{l^3} & & & & \frac{-6EJ_\zeta}{l^2} \\
 & & \frac{-12EJ_\eta}{l^3} & & \frac{6EJ_\eta}{l^2} & & & & \frac{12EJ_\eta}{l^3} & & \frac{6EJ_\eta}{l^2} & \\
 & & & \frac{-GJ_s}{l} & & & & & & \frac{GJ_s}{l} & & \\
 & & \frac{-6EJ_\eta}{l^2} & & \frac{2EJ_\eta}{l} & & & & \frac{6EJ_\eta}{l^2} & & \frac{4EJ_\eta}{l} & \\
 & \frac{6EJ_\zeta}{l^2} & & & & \frac{2EJ_\zeta}{l} & & \frac{-6EJ_\zeta}{l^2} & & & & \frac{4EJ_\zeta}{l}
\end{bmatrix}$$

3D FRAME ELEMENT



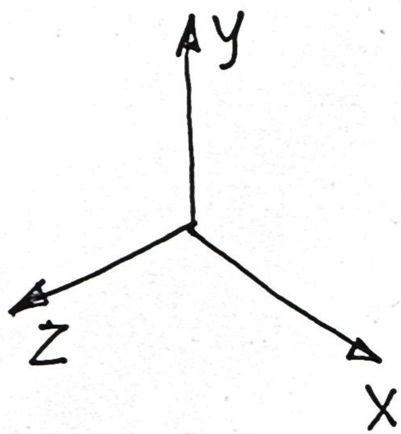
LOCAL PARAMETERS IN THE COORDINATE SYSTEM $\xi\eta\zeta$

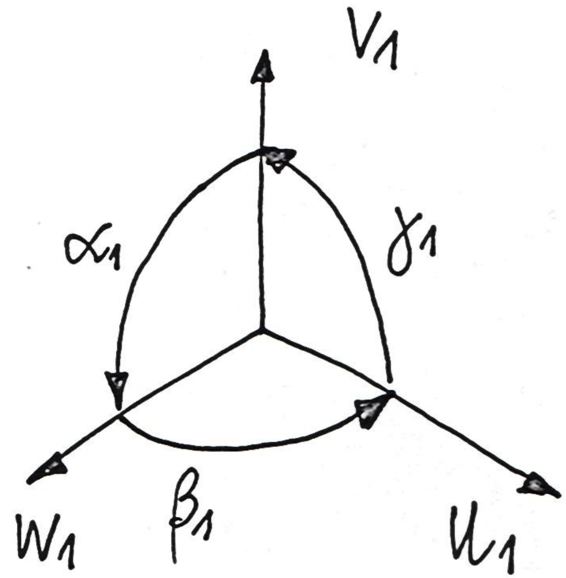
$$\underset{1\times 12}{\lfloor q \rfloor_e} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12} \rfloor$$



Local parameters in the coordinate system xyz

$$\lfloor q_g \rfloor_{e \atop 1\times 12} = \lfloor u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2 \rfloor$$





$$U_e = \frac{1}{2} \underset{1\times12}{\lfloor q \rfloor_e} \; \underset{12\times12}{[k]_e} \; \underset{12\times1}{\{q\}_e} \qquad \text{where :}$$

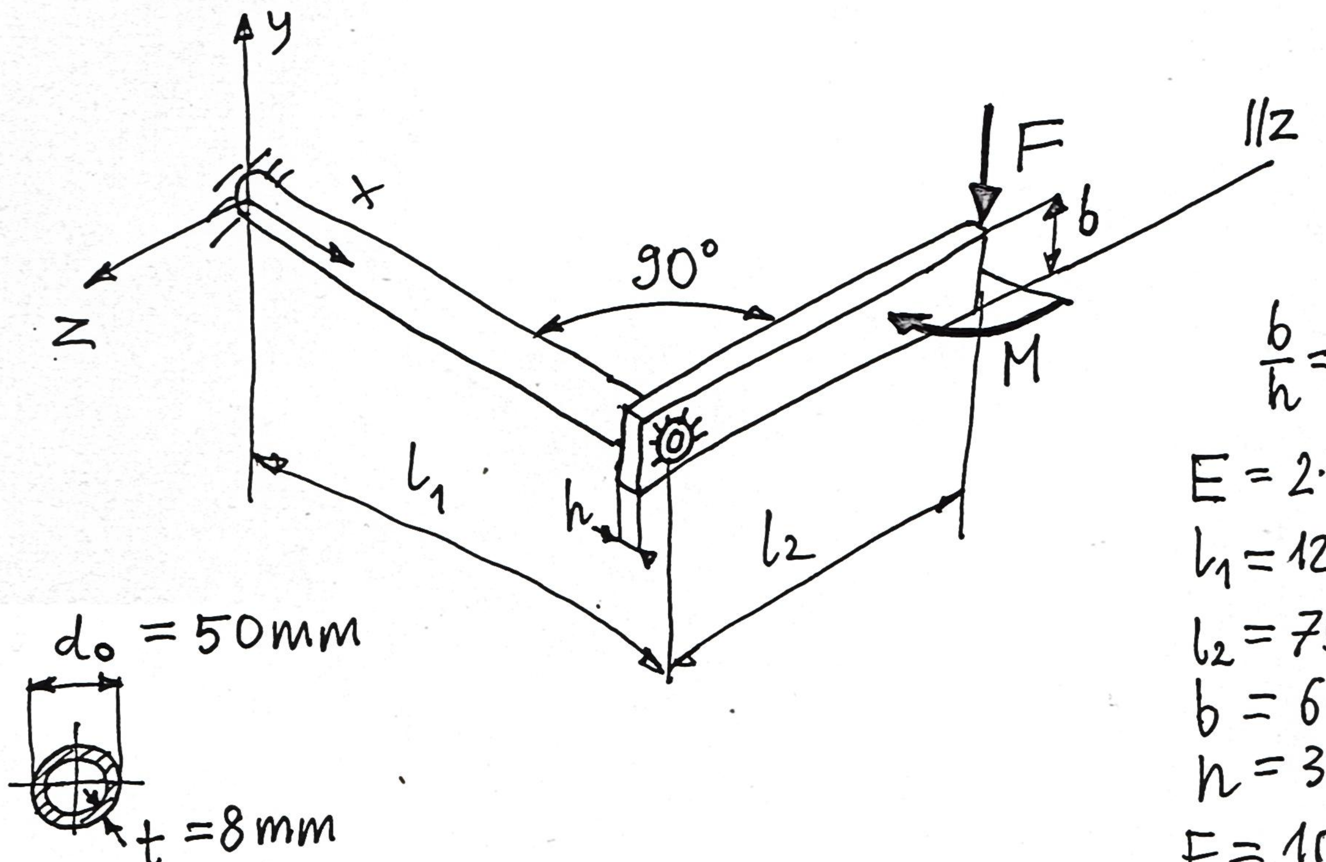
$$[k]_e = \begin{bmatrix}
a & & & & & & -a & & & & & \\
& b & & d & & & & -b & & d & & \\
& & c & & e & & & & -c & & e & \\
& d & & 2r & & & & -d & & r & & \\
& & e & & 2s & & & & -e & & s & \\
& & & & & t & & & & & & -t \\
-a & & & & & & a & & & & & \\
& -b & & -d & & & & b & & -d & & \\
& & -c & & -e & & & & c & & -e & \\
& d & & r & & & & -d & & 2r & & \\
& & e & & s & & & & -e & & 2s & \\
& & & & & -t & & & & & & t
\end{bmatrix}$$

$$a = \frac{EA}{L}, \quad b = \frac{12EJ_\zeta}{l_e^3}, \quad c = \frac{12EJ_\eta}{l_e^3}, \quad d = \frac{6EJ_\zeta}{l_e^2},$$

$$e = \frac{6EJ_\eta}{l_e^2}, \quad r = \frac{2EJ_\zeta}{l_e}, \quad s = \frac{2EJ_\eta}{l_e}, \quad t = \frac{G \cdot J_0}{l_e}$$

$$U_e = \frac{1}{2} \underset{1\times12}{\lfloor q_g \rfloor_e} \cdot \underbrace{\underset{12\times12}{[T_f]^T} \cdot \underset{12\times12}{[k]_e} \cdot \underset{12\times12}{[T_f]}}_{\underset{12\times12}{[kg]_e}} \cdot \underset{12\times1}{\{q_g\}_e}$$

EXAMPLE : BUILD A FE MODEL USING 3D FRAME ELEMENTS. FIND UNKNOWN DISPLACEMENTS, STRESSES AND REACTIONS.



$\frac{b}{h} = 2$

$E = 2 \cdot 10^5\,MPa$

$l_1 = 1200\,mm$

$l_2 = 750\,mm$

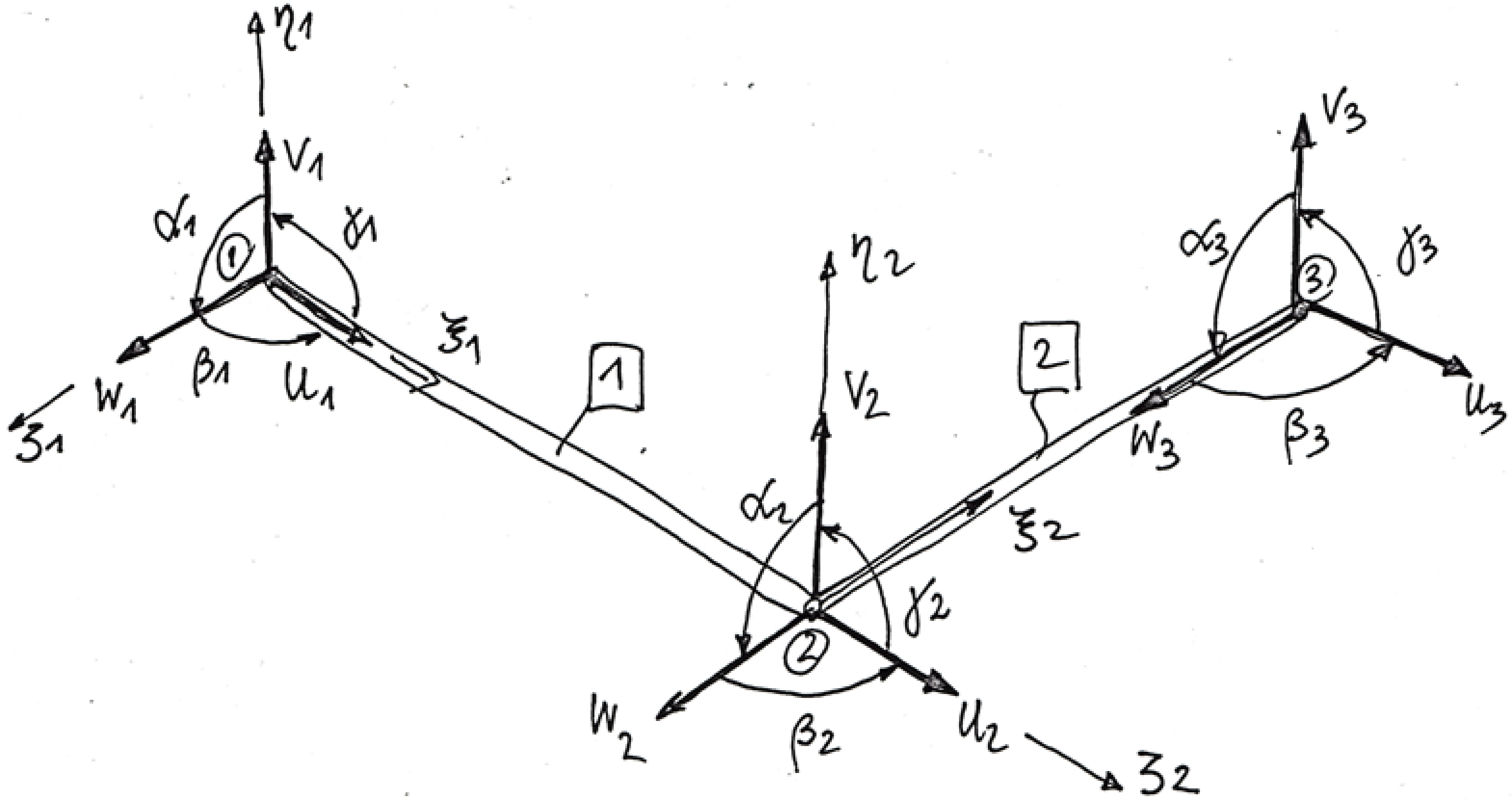
$b = 60\,mm$

$h = 30\,mm$

$F = 1000\,N$

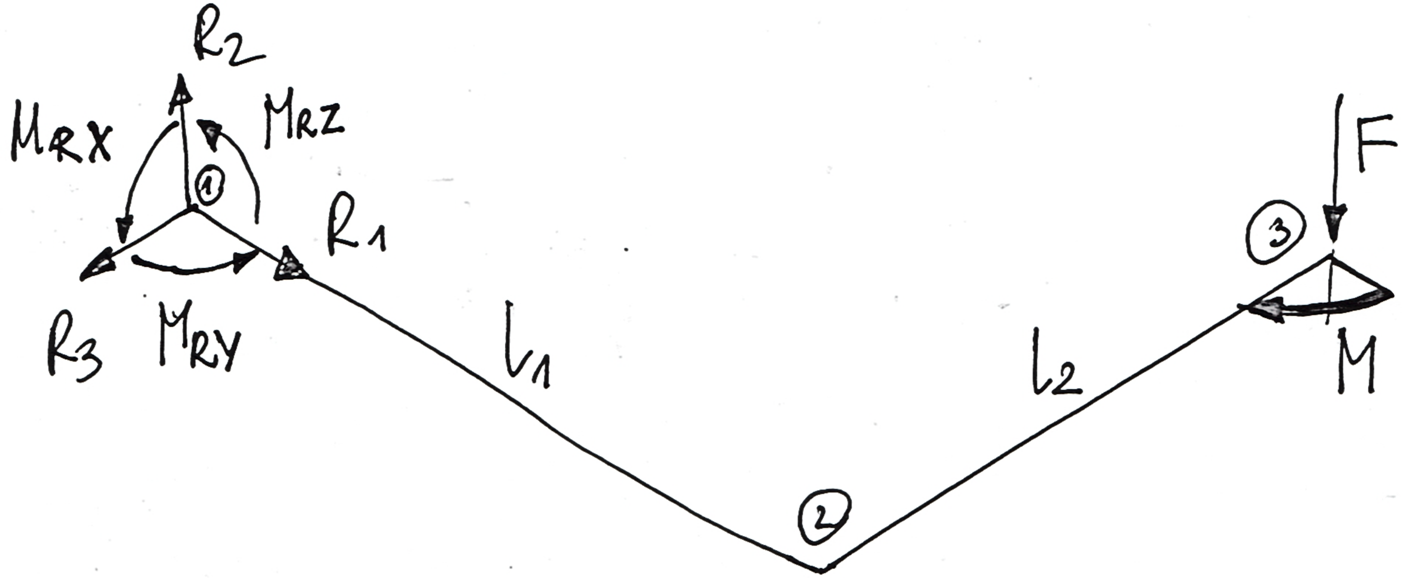
$M = 1000000\,Nmm$

Global parameters in the coordinate system xyz



$$\underset{1\times 18}{\lfloor q \rfloor} = \lfloor u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2, u_3, v_3, w_3, \alpha_3, \beta_3, \gamma_3 \rfloor$$

Global load vector



$$\lfloor F \rfloor_{1\times18} = \lfloor R_1, R_2, R_3, M_{RX}, M_{RY}, M_{RZ}, 0, 0, 0, 0, 0, 0, 0, -F, 0, 0, -M, 0 \rfloor$$

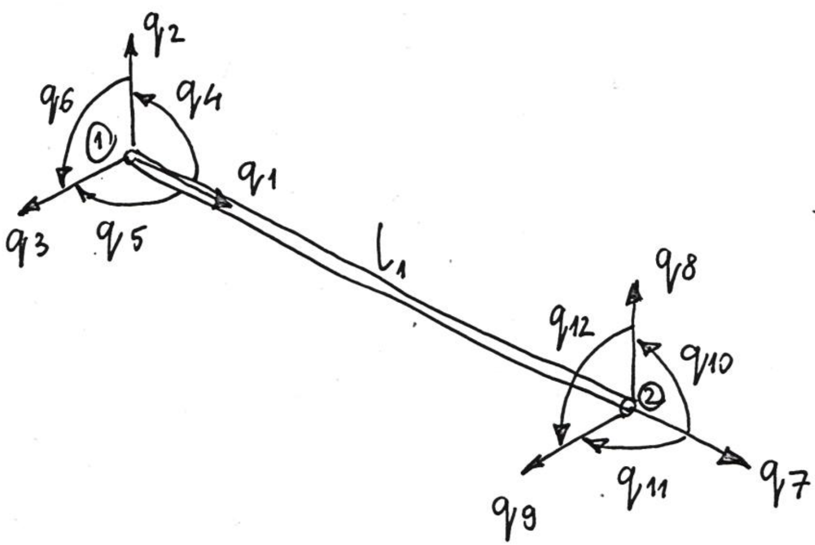
ELEMENT $\boxed{1}$:

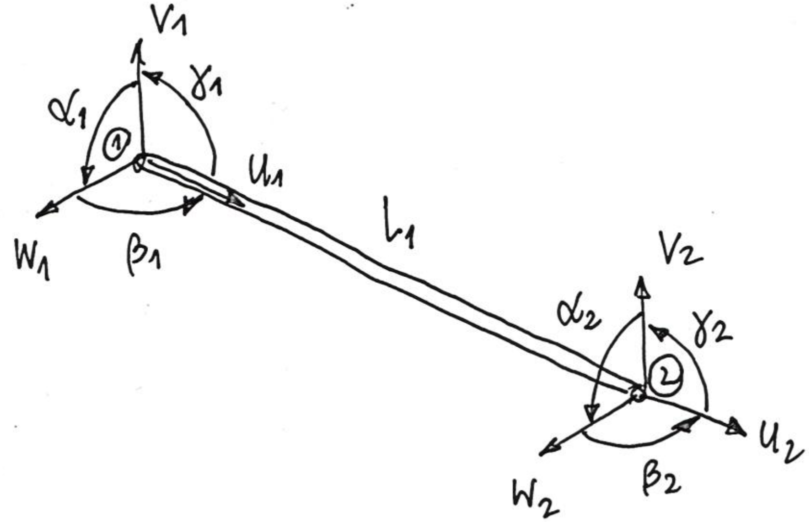
$$\underset{1\times 12}{\lfloor q \rfloor_1} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12} \rfloor$$

$$\underset{1\times 12}{\lfloor q_g \rfloor_1} = \lfloor u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2 \rfloor$$

$$q_1 = u_1, \quad q_2 = v_1, \quad q_3 = w_1, \quad q_4 = \gamma_1, \quad q_5 = -\beta_1, \quad q_6 = \alpha_1$$

$$q_7 = u_2, \quad q_8 = v_2, \quad q_9 = w_2, \quad q_{10} = \gamma_2, \quad q_{11} = -\beta_2, \quad q_{12} = \alpha_2$$





$$\underset{12\times 1}{\{q\}_1} = \underset{12\times 12}{[T_f]_1} \cdot \underset{12\times 1}{\{q_g\}_1}$$

$$\underset{12\times 12}{[T_f]_1} = \left[\begin{array}{cccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & \\
0 & 1 & 0 & 0 & 0 & 0 & \\
0 & 0 & 1 & 0 & 0 & 0 & \underset{6\times 6}{[0]} \\
0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & -1 & 0 & \\
0 & 0 & 0 & 1 & 0 & 0 & \\
\hline
 & & & & & & \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{matrix} \\
 & & & \underset{6\times 6}{[0]} & & & \\
\end{array}\right] = [T_f]^T$$

$$\underset{12\times12}{[k_g]_1} = \underset{12\times12}{[T_f]_1^T} \cdot \underset{12\times12}{[k]_1} \cdot \underset{12\times12}{[T_f]_1}$$

$$a_1 = \frac{EA_1}{l_1}, \quad b_1 = \frac{12\,EJ_{\zeta 1}}{l_1^3}, \quad c_1 = b_1, \quad d_1 = \frac{6\,EJ_{\zeta 1}}{l_1^2}, \quad e_1 = d_1,$$

$$r_1 = \frac{2\,EJ_{\zeta 1}}{l_1}, \quad s_1 = r_1, \quad t_1 = \frac{G\,J_{o1}}{l_1}, \quad A_1 = \frac{\pi\left(d_o^2 - (d_o - 2t)^2\right)}{4}$$

$$J_{\zeta 1} = J_{\eta 1} = \frac{\pi}{64}\left(d_o^4 - (d_o - 2t)^4\right), \quad J_{o1} = 2 \cdot J_{\zeta 1}$$

$$\underset{18\times18}{[k_g]_1^*} = \left[\begin{array}{c|c} [k_g]_1 & \underset{12\times6}{[0]} \\ \hline \underset{6\times12}{[0]} & \underset{6\times6}{[0]} \end{array}\right]$$

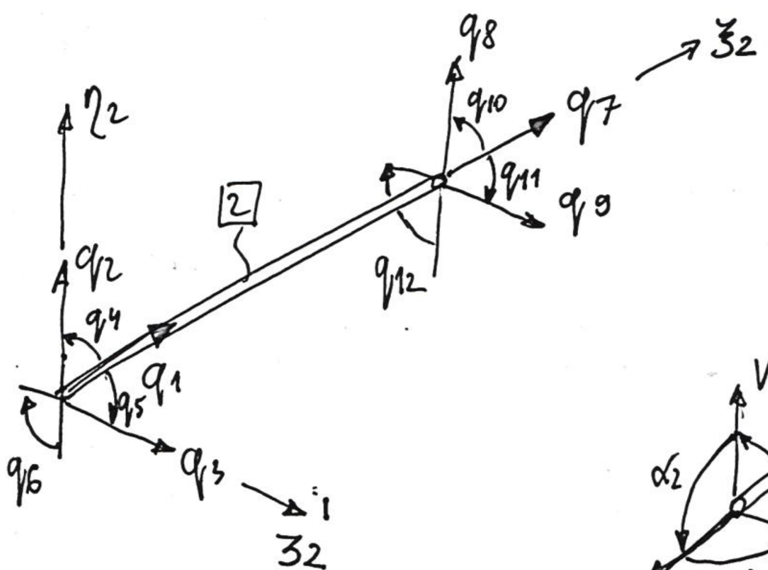
ELEMENT $\boxed{2}$

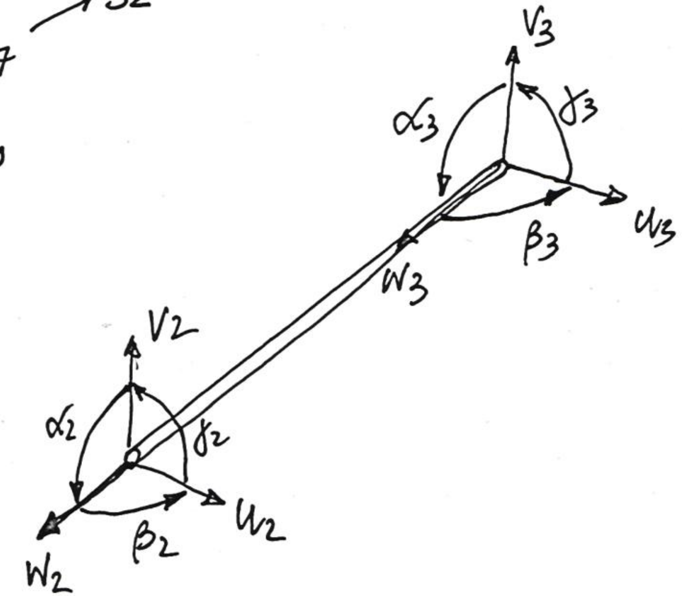
$$\lfloor q \rfloor_2 = \lfloor q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12} \rfloor$$

$$\lfloor q_g \rfloor_2 = \lfloor u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2, u_3, v_3, w_3, \alpha_3, \beta_3, \gamma_3 \rfloor$$

$$q_1 = -w_2, \quad q_2 = v_2, \quad q_3 = u_2, \quad q_4 = \alpha_2, \quad q_5 = -\beta_2, \quad q_6 = -\gamma_2$$

$$q_7 = -w_3, \quad q_8 = v_3, \quad q_9 = u_3, \quad q_{10} = \alpha_3, \quad q_{11} = -\beta_3, \quad q_{12} = -\gamma_3$$





$$\underset{12\times1}{\{q\}_2} = \underset{12\times12}{[T_f]_2} \cdot \underset{12\times1}{\{q_g\}_2}$$

$$[T_f]_2 = \left[\begin{array}{cccccccccccc}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\multicolumn{6}{c}{} & 0 & 0 & -1 & 0 & 0 & 0 \\
\multicolumn{6}{c}{} & 0 & 1 & 0 & 0 & 0 & 0 \\
\multicolumn{6}{c}{\underset{6\times6}{[0]}} & 1 & 0 & 0 & 0 & 0 & 0 \\
\multicolumn{6}{c}{} & 0 & 0 & 0 & 1 & 0 & 0 \\
\multicolumn{6}{c}{} & 0 & 0 & 0 & 0 & -1 & 0 \\
\multicolumn{6}{c}{} & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right]$$

$$\underset{12\times 12}{[k_g]_2} = \underset{12\times 12}{[T_f]_2^T} \cdot \underset{12\times 12}{[k]_2} \cdot \underset{12\times 12}{[T_f]_2}$$

$$a_2 = \frac{EA_2}{l_2}, \quad b_2 = \frac{12EJ_{\zeta 2}}{l_2^3}, \quad c_2 = \frac{12EJ_{\eta 2}}{l_2^3}, \quad d_2 = \frac{6EJ_{\zeta 2}}{l_2^2}$$

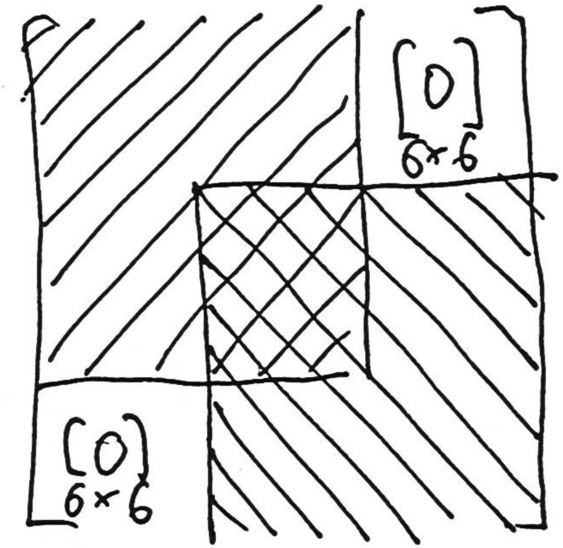
$$e_2 = \frac{6EJ_{\eta 2}}{l_2^2}, \quad r_2 = \frac{2EJ_{\zeta 2}}{l_2}, \quad s_2 = \frac{2EJ_{\eta 2}}{l_2}, \quad t_2 = \frac{G \cdot J_{o2}}{l_2}$$

$$J_{\zeta 2} = \frac{hb^3}{12}, \quad J_{\eta 2} = \frac{bh^3}{12}, \quad J_{o2} = 0.457\, bh^3$$

$$A_2 = b \cdot h$$

$$\underset{18\times 18}{[k_g]_2^*} = \left[\begin{array}{c|c} \underset{6\times 6}{[0]} & \underset{6\times 12}{[0]} \\ \hline \underset{12\times 6}{[0]} & [k_g]_2 \end{array}\right]$$

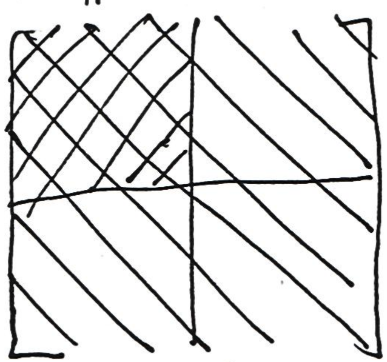
$$\underset{18\times 18}{[K]} = \underset{18\times 18}{[K_g]_1^{*}} + \underset{18\times 18}{[K_g]_2^{*}} =$$



$$\underset{18\times18}{[K]}\cdot\underset{18\times1}{\{q\}}=\underset{18\times1}{\{F\}}$$

$u_1=0,\ v_1=0,\ w_1=0,\ \alpha_1=0,\ \beta_1=0,\ \gamma_1=0$

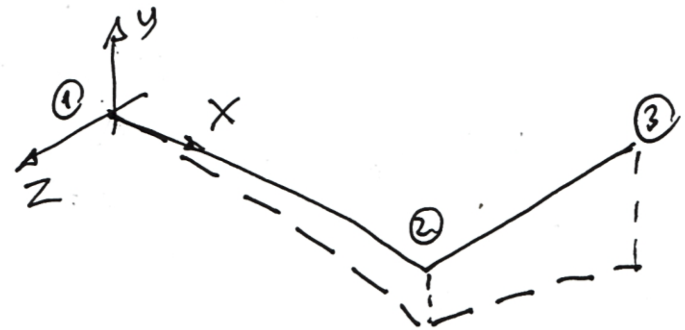
$$\underset{12\times12}{[K]}\cdot\underset{12\times1}{\{q\}}=\underset{12\times1}{\{F\}} \Rightarrow \underset{12\times1}{\{q\}}=\underset{12\times12}{[K]^{-1}}\cdot\underset{12\times1}{\{F\}}$$

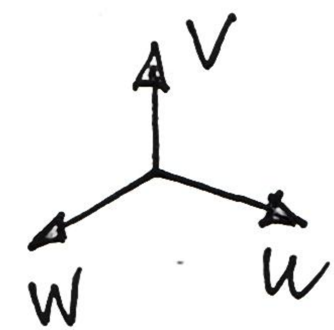


$$\underset{18\times18}{[K]}\cdot\underset{18\times1}{\{q\}}=\underset{18\times1}{\{F\}} \Rightarrow \text{REACTIONS}$$

DOF SOLUTION

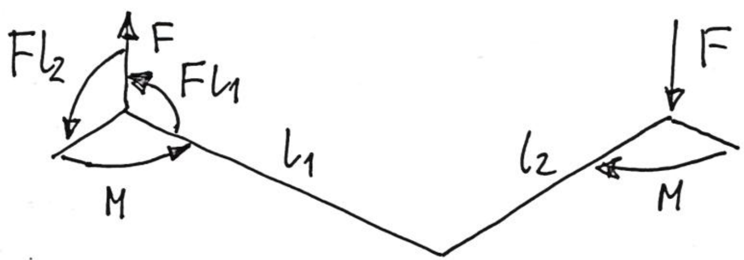
$$\{q\}_{12\times 1} = \begin{Bmatrix} u_2 \\ v_2 \\ w_2 \\ \alpha_2 \\ \beta_2 \\ \gamma_2 \\ u_3 \\ v_3 \\ w_3 \\ \alpha_3 \\ \beta_3 \\ \gamma_3 \end{Bmatrix} = \begin{Bmatrix} 0\ \text{mm} \\ -11.94\ \text{mm} \\ 14.93\ \text{mm} \\ -0.0243\ \text{rad} \\ -0.0249\ \text{rad} \\ -0.015\ \text{rad} \\ 29.07\ \text{mm} \\ -31.43\ \text{mm} \\ 14.93\ \text{mm} \\ -0.0269\ \text{rad} \\ -0.0527\ \text{rad} \\ -0.015\ \text{rad} \end{Bmatrix}$$





REACTIONS

$$\underset{18\times1}{\{F\}} = \underset{18\times18}{[K]} \cdot \underset{18\times1}{\{q\}} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ M_{RX} \\ M_{RY} \\ M_{RZ} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -F \\ 0 \\ 0 \\ -M \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1000\,N \;\; (+F) \\ 0 \\ 750000 \;\; (F \cdot l_2) \\ 1000000 \;\; (M) \\ 1200000 \;\; (F l_1) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -F \\ 0 \\ 0 \\ -M \\ 0 \end{Bmatrix}$$

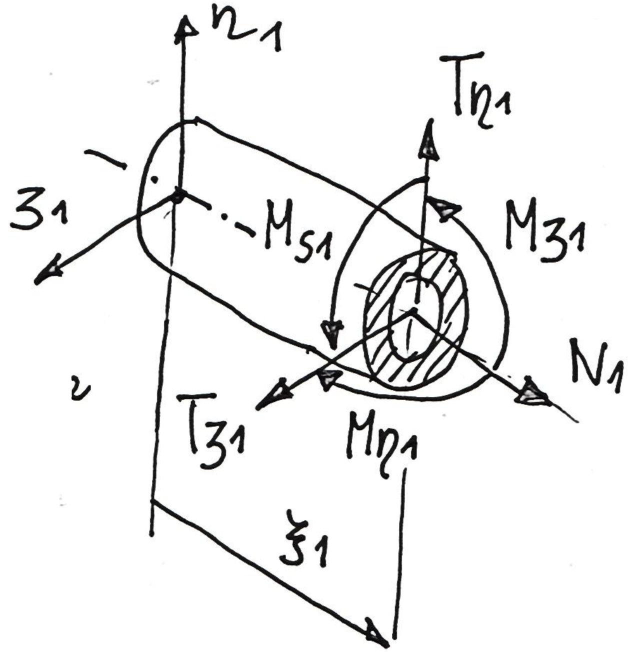


THE STRUCTURE IS IN EQUILIBRIUM

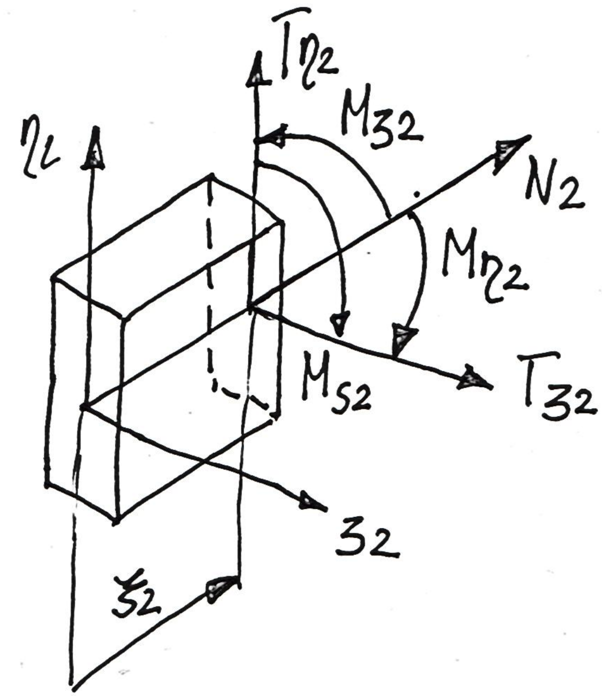
ELEMENT SOLUTION

$$\underset{12\times 1}{\{q\}_i} = \underset{12\times 12}{[T_f]_i} \cdot \underset{12\times 1}{\{q_g\}_i} \quad , \quad i = 1, 2$$

$\boxed{1}$



$\boxed{2}$



AXIAL BAR: $(q_1, q_7)_i$

$$\varepsilon_{\xi i} = \frac{(q_7 - q_1)_i}{l_i} \quad , \quad \sigma_{\xi i} = E \cdot \varepsilon_{\xi i} \quad , \quad N_i = \sigma_{\xi i} A_i$$

BEAM:

I) BENDING IN $(\xi\eta)_i$ PLANE: $(q_2, q_4, q_8, q_{10})_i$

$$v_i(\xi_i) = \underset{1\times 4}{\lfloor N(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_2 \\ q_4 \\ q_8 \\ q_{10} \end{Bmatrix}_i$$

$$M_{zi}(\xi_i) = E \cdot J_{zi} \cdot v_i''(\xi_i) = $$
$$= E \cdot J_{zi} \cdot \underset{1\times 4}{\lfloor N''(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_2 \\ q_4 \\ q_8 \\ q_{10} \end{Bmatrix}_i$$

$$\sigma_{\xi i}^{I} = -\frac{M_{zi}(\xi_i)\cdot\eta_i}{J_{zi}}$$

$$T_{\eta i}(\xi_i) = -EJ_{zi}\cdot v_i'''(\xi_i) =$$

$$= -EJ_{zi}\cdot\underset{1\times 4}{\left[N'''\right]}\cdot\begin{Bmatrix} q_2\\ q_4\\ q_8\\ q_{10}\end{Bmatrix}_i = \text{const}$$

SHEAR STRESS CAUSED BY $T_{\eta i}(\xi_i)$ IS NEGLECTED.

II) BENDING IN $(\xi\zeta)$ PLANE : $(q_3, q_5, q_9, q_{11})_i$

$$w_i(\xi_i) = \underset{1\times 4}{\lfloor N(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_3 \\ q_5 \\ q_9 \\ q_{11} \end{Bmatrix}_i$$

$$M_{\eta i}(\xi_i) = E \cdot J_{\eta i}\, w_i''(\xi_i) =$$

$$= EJ_{\eta i} \cdot \underset{1\times 4}{\lfloor N''(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_3 \\ q_5 \\ q_9 \\ q_{11} \end{Bmatrix}_i$$

$$\sigma_{\xi i}^{II} = - \frac{M_{\eta i}(\xi_i)\cdot \zeta_i}{J_{\eta i}}$$

$$T_{\zeta i}(\xi_i) = - E J_{\eta i} \cdot w_i'''(\xi_i) =$$

$$= - E J_{\eta i} \lfloor N''' \rfloor_{1\times 4} \cdot \begin{Bmatrix} q_3 \\ q_5 \\ q_9 \\ q_{11} \end{Bmatrix}_i = \text{const}$$

SHEAR STRESS CAUSED BY $T_{\zeta i}(\xi_i)$ IS NEGLECTED.

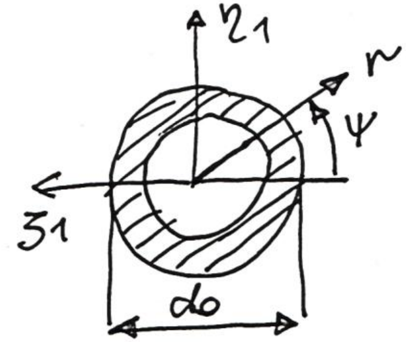
TORSION BAR: $(q_6, q_{12})_i$

$$\varphi_i(\xi_i) = \underset{1\times 2}{\lfloor N(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_6 \\ q_{12} \end{Bmatrix}_i = \left\lfloor 1 - \frac{\xi_i}{l_i}, \ \frac{\xi_i}{l_i} \right\rfloor \cdot \begin{Bmatrix} q_6 \\ q_{12} \end{Bmatrix}_i =$$

$$= (q_6)_i + \frac{(q_{12} - q_6)_i}{l_i} \cdot \xi_i$$

$$\begin{Bmatrix} q_6 \\ q_{12} \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

ELEMENT $\boxed{1}$:



$$\tau_1(r) = G \cdot \gamma_1(r) = \frac{E}{2(1+\nu)} \cdot \frac{d\varphi_1(\xi_1)}{d\xi_1} \cdot r =$$

$$= \frac{E}{2(1+\nu)} \cdot \left\lfloor -\frac{1}{L_1} , \frac{1}{L_1} \right\rfloor \cdot \begin{Bmatrix} q_6 \\ q_{12} \end{Bmatrix}_1 \cdot r =$$

$$= \frac{E\,(q_{12} - q_6)_1}{2(1+\nu)\,L_1} \cdot r \quad 12 \quad , (q_6)_1 = 0$$

$$\tau_{1\,max} = \tau_1\left(\frac{d_0}{2}\right) = \frac{E \cdot (q_{12})_1 d_0}{4(1+\nu)\,L_1} = \frac{E\, d_0\, \alpha_2}{4(1+\nu) L_1}$$

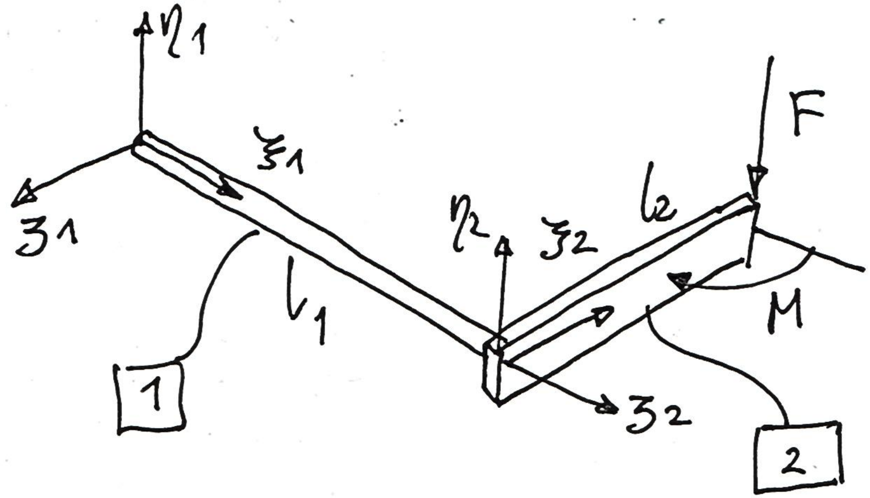
$$M_{S1} = \frac{\tau_1(r) \cdot J_{01}}{r} = \frac{E \cdot (q_{12})_1 \cdot J_{01}}{2(1+\nu)\,L_1} = \frac{E \alpha_2 J_{01}}{2(1+\nu) L_1} = \text{const}$$

ELEMENT $\boxed{2}$:

$$\begin{Bmatrix} q_6 \\ q_{12} \end{Bmatrix}_2 = \begin{Bmatrix} \gamma_2 \\ \gamma_3 \end{Bmatrix} = \begin{Bmatrix} -0.015 \text{ rad} \\ -0.015 \text{ rad} \end{Bmatrix} \Rightarrow \varphi_2(\xi_2) = q_6 = \text{const}$$

$$\Rightarrow \frac{d\varphi_2(\xi_2)}{d\xi_2} = 0 \Rightarrow \begin{matrix} \tau_2 = 0 \\ M_{s2} = 0 \end{matrix}$$

ELEMENT RESULTS



NORMAL FORCES:

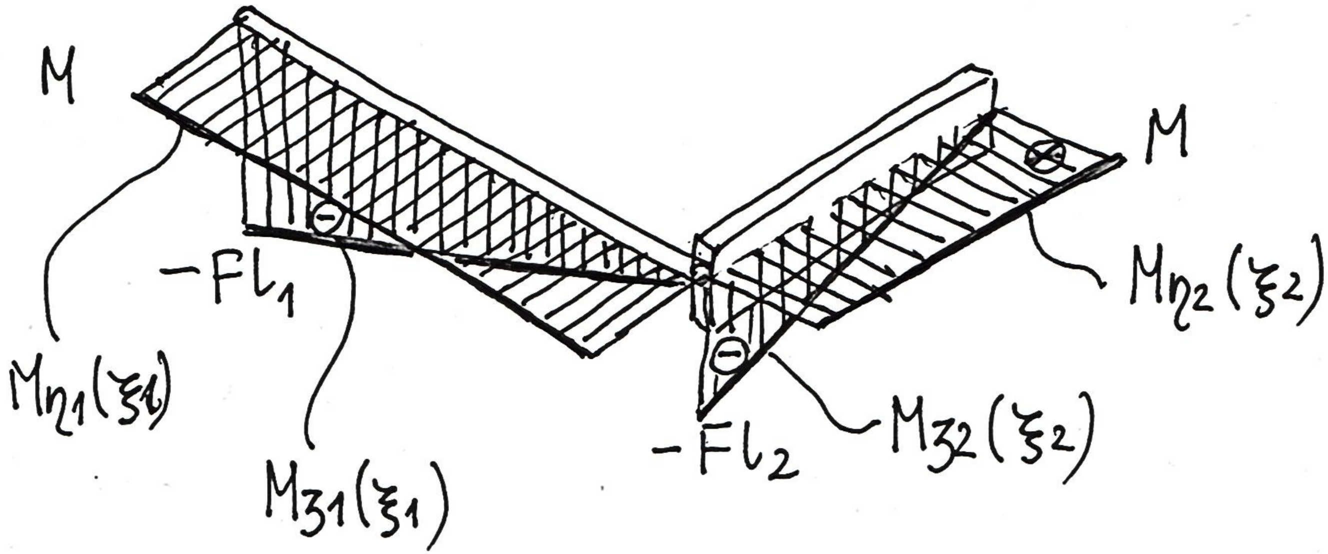
$$N_1 = 0 \quad , \quad N_2 = 0$$

SHEAR FORCES:

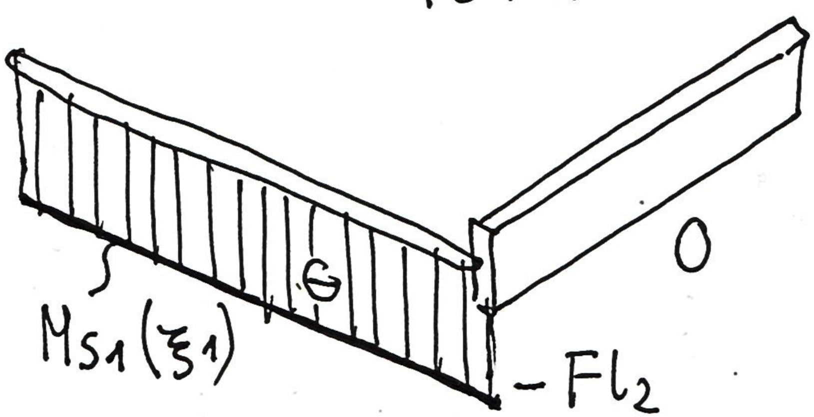
$$T_{\eta_1} = -F \quad , \quad T_{\eta_2} = -F$$

$$T_{\zeta_1} = 0 \quad , \quad T_{\zeta_2} = 0$$

BENDING MOMENTS :



TORQUE. :



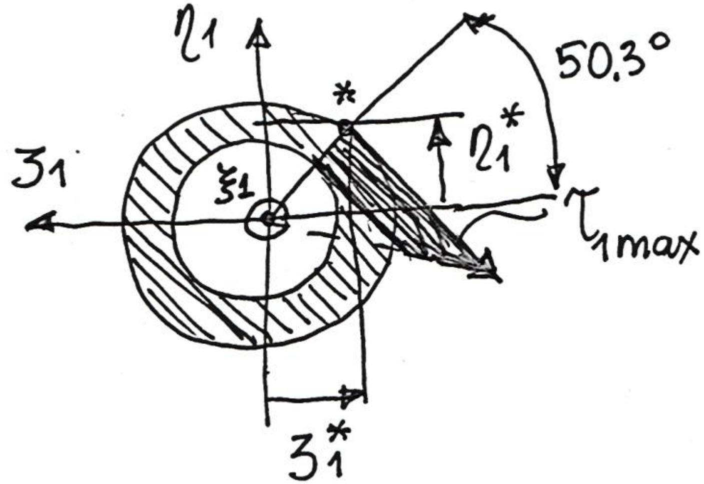
POINT OF THE HIGHEST STRESS :

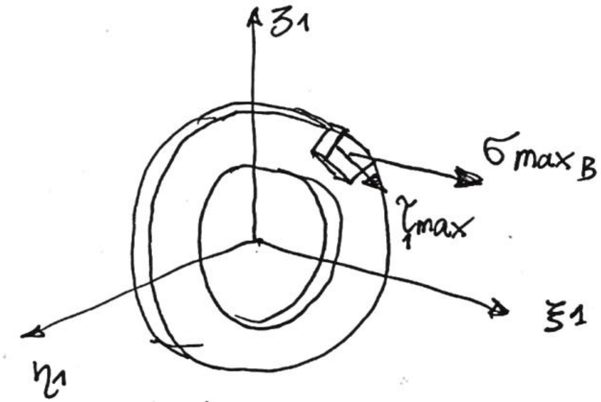
ELEMENT $\boxed{1}$, $\xi_1 = 0$

NORMAL STRESS DUE TO BENDING :

$$\sigma_{MAX\,B} = \sigma_{\xi_1}^{I} + \sigma_{\xi_1}^{II} =$$

$$= -\frac{M_{\zeta_1}(0)\cdot \eta_1^*}{J_{\zeta_1}} - \frac{M_{\eta_1}(0)\cdot \zeta_1^*}{J_{\eta_1}} = 161.9\,\mathrm{MPa}$$





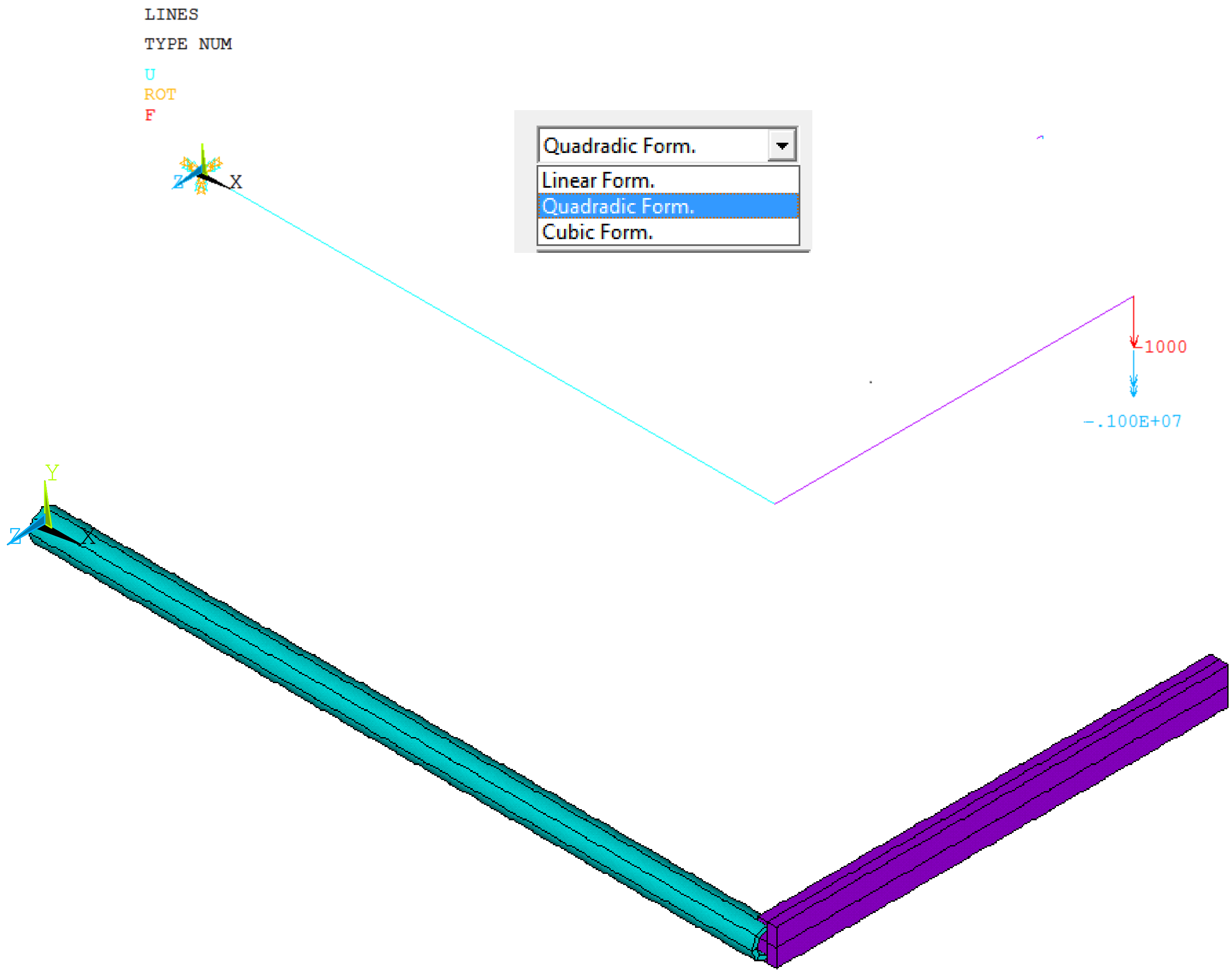
SHEAR STRESS

$$\tau_{1max} = \frac{E d_0 \alpha_2}{4(1+\nu)L_1} = -38.86 \text{ MPa}$$

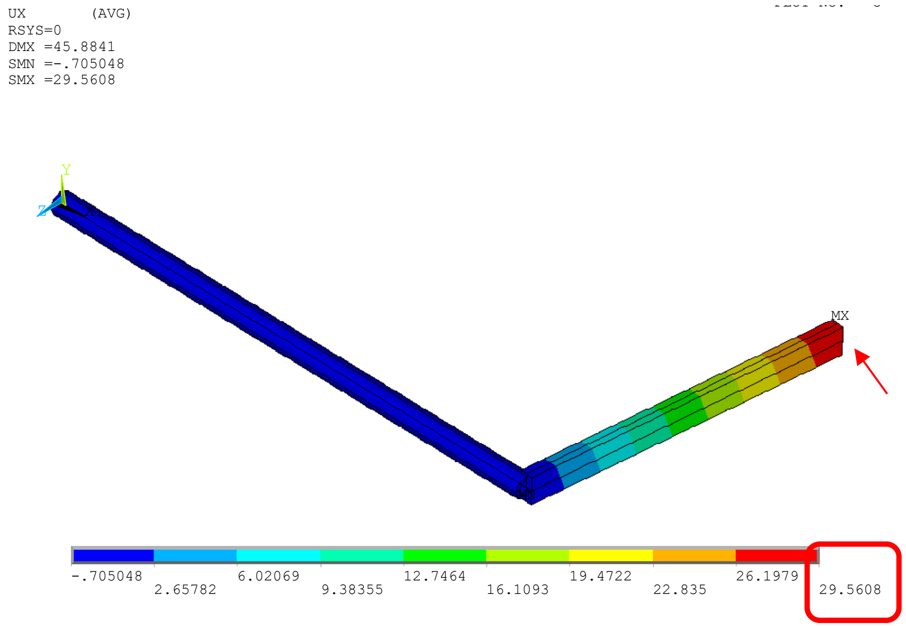
MAXIMUM EQUIVALENT STRESS :

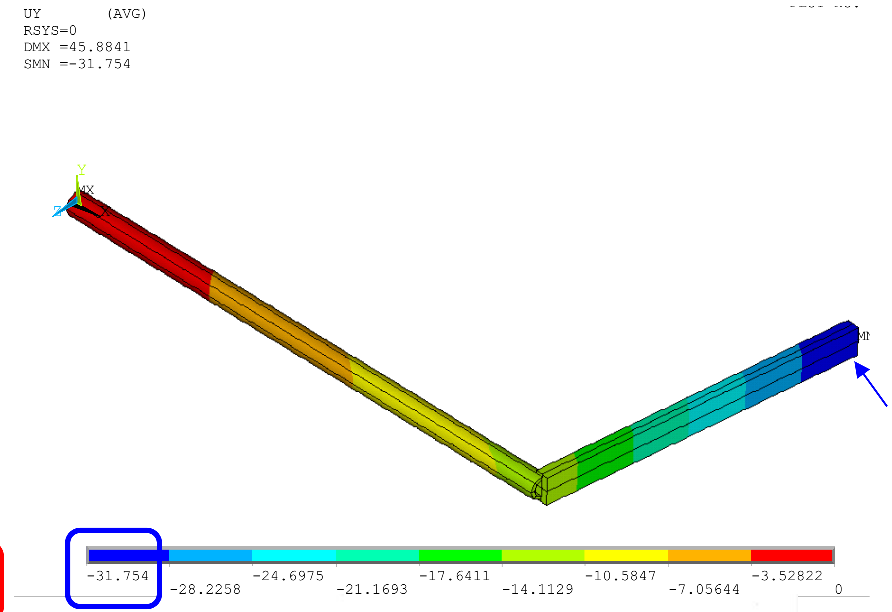
$$\sigma_{EQV} = \sqrt{\sigma_{MAX\,B}^2 + 3\tau_{1max}^2} = 175.34 \text{ MPa}$$

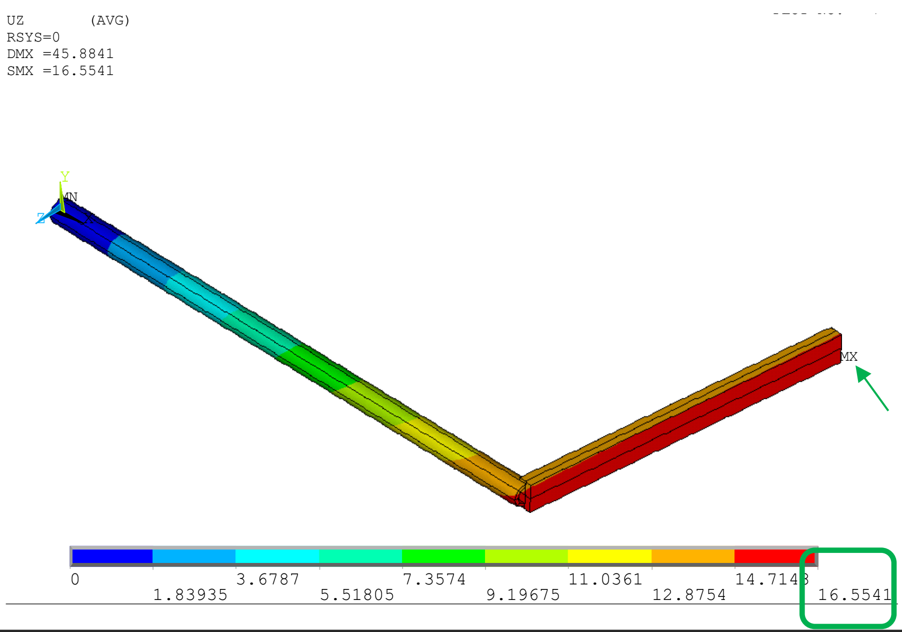
Two-element model (Ansys)



Two-element model (Ansys) – Displacements in X, Y and Z

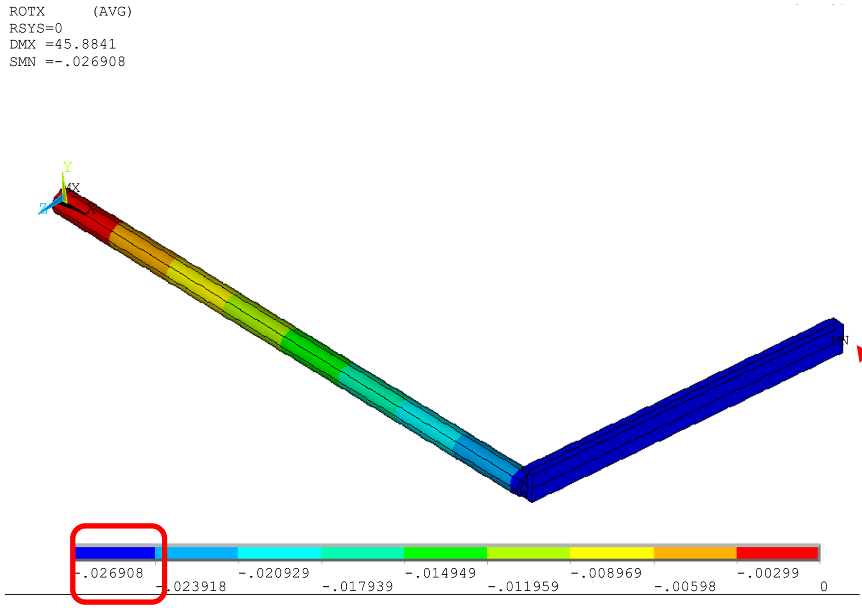


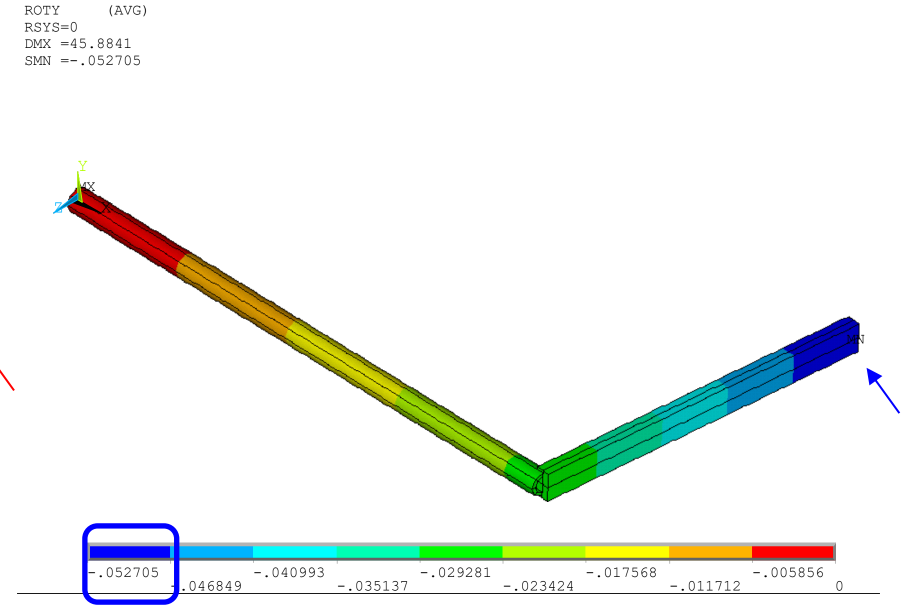


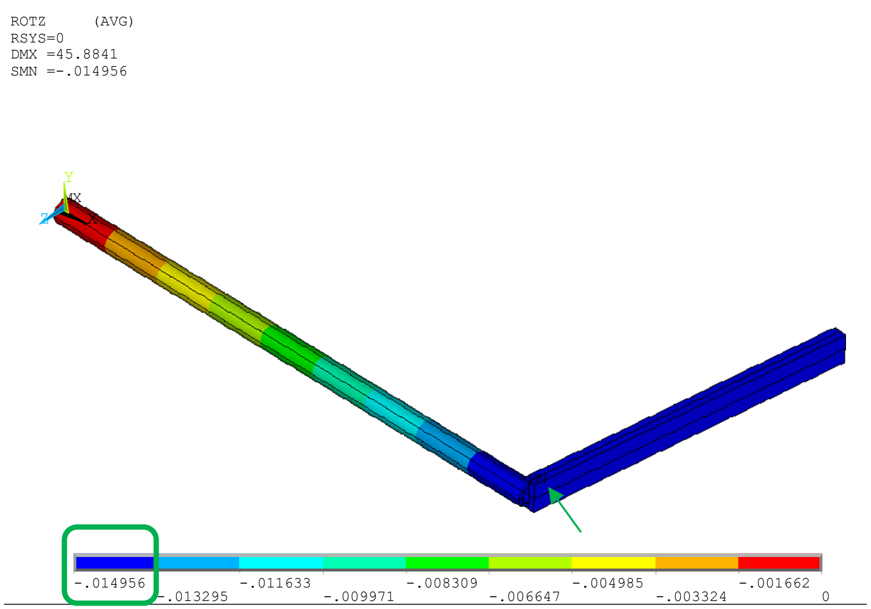


$$\underset{12\times 1}{\{q\}} = \begin{Bmatrix} u_2 \\ v_2 \\ w_2 \\ \alpha_2 \\ \beta_2 \\ \gamma_2 \\ u_3 \\ v_3 \\ w_3 \\ \alpha_3 \\ \beta_3 \\ \gamma_3 \end{Bmatrix} = \begin{Bmatrix} 0 \text{ mm} \\ -11.94 \text{ mm} \\ 14.93 \text{ mm} \\ -0.0243 \text{ rad} \\ -0.0249 \text{ rad} \\ -0.015 \text{ rad} \\ \boxed{29.07 \text{ mm}} \\ \boxed{-31.43 \text{ mm}} \\ \boxed{14.93 \text{ mm}} \\ -0.0269 \text{ rad} \\ -0.0527 \text{ rad} \\ -0.015 \text{ rad} \end{Bmatrix}$$

Two-element model (Ansys) - Angular displacements

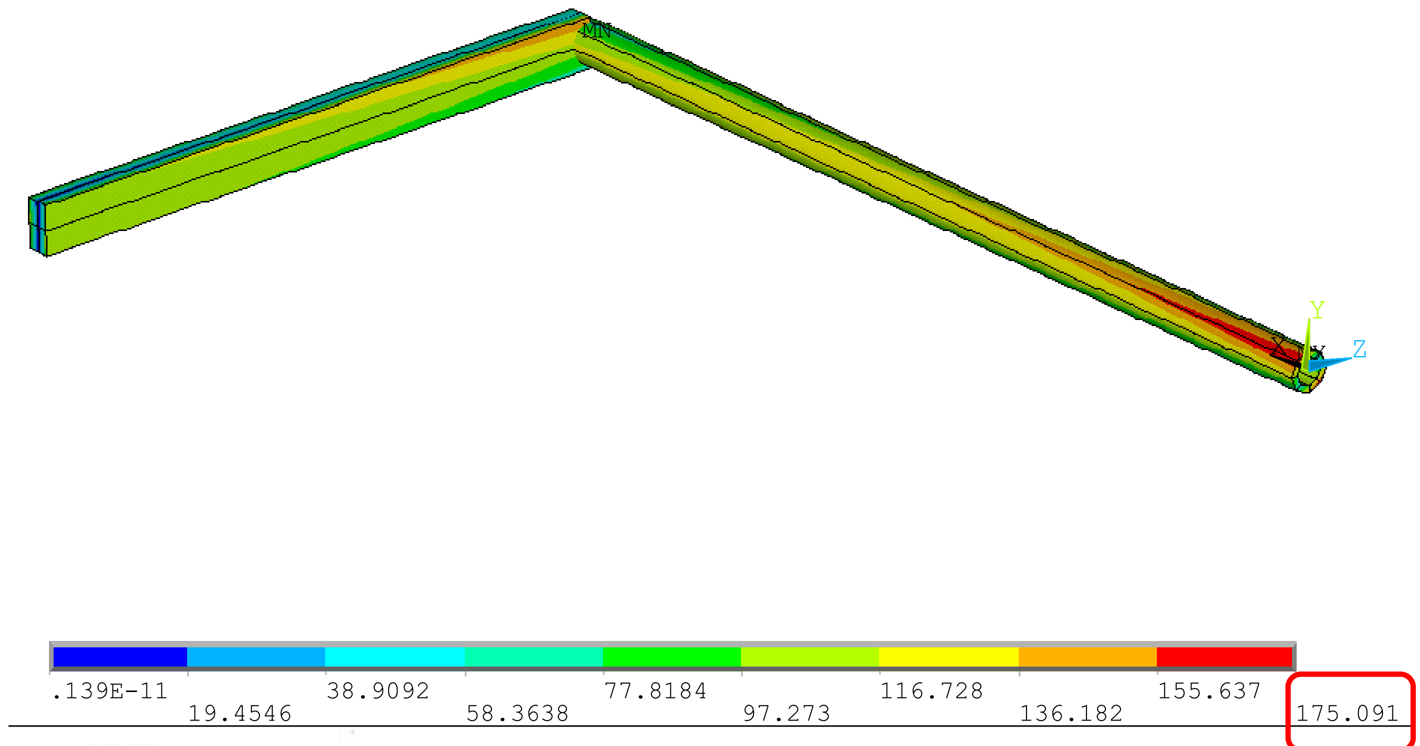


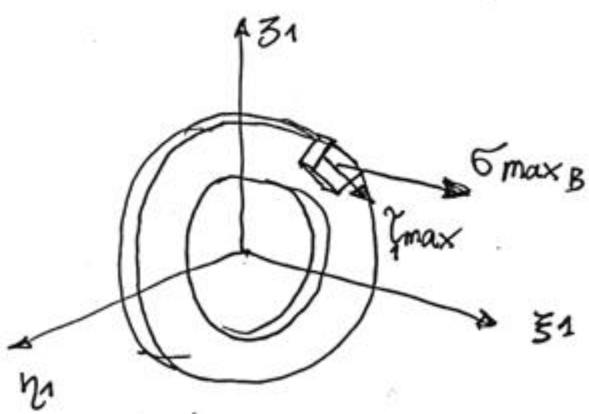




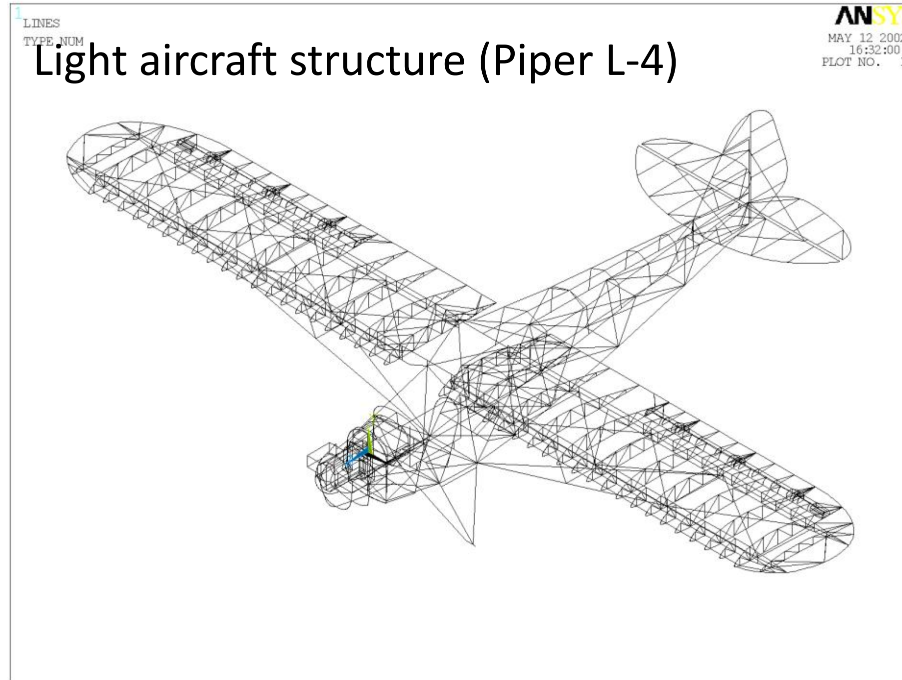
$$\underset{12\times 1}{\{q\}} = \begin{Bmatrix} u_2 \\ v_2 \\ w_2 \\ \alpha_2 \\ \beta_2 \\ \gamma_2 \\ u_3 \\ v_3 \\ w_3 \\ \alpha_3 \\ \beta_3 \\ \gamma_3 \end{Bmatrix} = \begin{Bmatrix} 0 \text{ mm} \\ -11.94 \text{ mm} \\ 14.93 \text{ mm} \\ -0.0243 \text{ rad} \\ -0.0249 \text{ rad} \\ \boxed{-0.015 \text{ rad}} \\ 29.07 \text{ mm} \\ -31.43 \text{ mm} \\ 14.93 \text{ mm} \\ \boxed{-0.0269 \text{ rad}} \\ \boxed{-0.0527 \text{ rad}} \\ -0.015 \text{ rad} \end{Bmatrix}$$

Two-element model (Ansys) - Von Mises stress (SEQV)

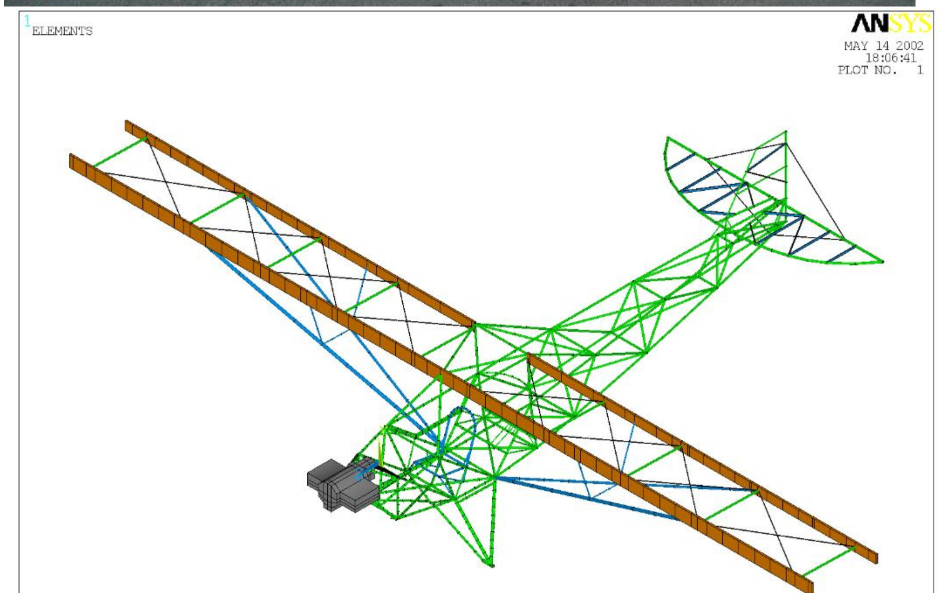




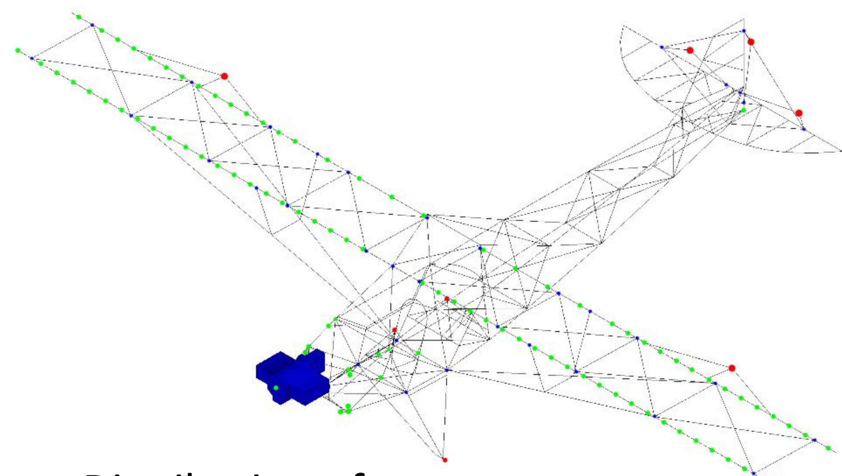
$$\sigma_{EQV} = \sqrt{\sigma_{MAX\,B}^{2} + 3\,\tau_{1\,max}^{2}} = \boxed{175.34\ MPa}$$



Light aircraft structure (Piper L-4)



FEM beam model



Distribution of concentrated masses in the substitute model